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Forced Periodic Oscillations and the Jones Polynomial

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We show that forced periodic oscillations in a nonlinear damped oscillator can be classified by means of the Jones (knot) polynomial; this is done by associating to any periodic oscillation a braid. We also discuss the relation of this approach to (Lie-point) symmetry analysis of the associated differential equations.

1. INTRODUCTION

Knot theory (Reidemeister, 1948; Burde and Zieschang, 1986; Rolfsen, 1976; Kauffman, 1983, 1987) is a classical subject in topology with—quite interestingly—physical motivations at the origins of its development at the end of the 19th century [the atomic theory of Kelvin; see also Atiyah (1990)].

In the 1920s, Alexander (1928) introduced his polynomial invariant as a powerful tool to study knots; actually, this approach was to remain unsurpassed until very recently when, in a burst of development, Jones and then others introduced a series of more powerful one- and two-variable invariant polynomials (Jones, 1985, 1986, 1987; Freyd *et al.*, 1985; Kauffman, 1988, 1989). Quite surprisingly, these new developments were related to apparently completely different subjects, and in particular physical ones.

Once it was realized that there is a close relation between knot theory and modern theoretical physics (Yang and Ge, 1989; Kauffman, 1990, and to appear; Witten, 1989*a*,*b*; Lusanna, 1990; Frolich and King, 1989; Wadati *et al.*, 1989; Jimbo, 1989), the developments followed at quite an impressive rhythm; here we just mention those concerned with statistical mechanics,

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quantum field theory, and integrable systems (Birman and Williams, 1983*a*,*b*).

It is not surprising, at this point, to think that there is a relation among knots and dynamical systems. This was pioneered by Birman and Williams (1983*a*,*b*) in connection with the Lorenz equation; the subject was further developed by Holmes and Williams (1985), also in relation to chaotic dynamics in continuous dynamical systems (ODE) (Holmes, 1986, 1988). This kind of approach was also pursued in the study of discrete dynamical systems, i.e., maps (Mielke, 1990).

The simple approach presented in this paper differs from the above ones in that we focus on regular dynamics, i.e., periodic orbits, rather than on chaotic ones (we also hope that an understanding of this situation can be of use in the study of the more complex cases).

We think that knot-theoretic methods can be of use also in this simple situation, and will indeed show that for two-dimensional dynamical systems possessing a natural time scale—e.g., a periodically forced oscillator—there is a canonical way to associate to each periodic orbit a braid and therefore a knot (we assume the reader has some acquaintance with the basics of knot theory; for a simple introduction see, e.g., Reidemeister (1948), Burde and Zeischang (1986), Rolfsen (1976), Kauffman (1983, 1987, 1988, 1990), Jones (1985, 1986, 1987), and Freyd *et al.* (1985).

We stress that in our approach knots are not associated to *sets* of orbits, but to *single* ones [in this respect, it is more similar to that of Mielke (1990)]; the presence of a time scale—period of the external forcing in this case, one-step in the discrete case, unperturbed period for nonlinear oscillators, etc.—is necessary, as will be clear in the following, for canonically mapping a two-dimensional dynamical system defined in $R^2 \times R^1$ (R^1 corresponds to the time coordinate) into the solid torus $R^2 \times S^1$ by a "period map" on the t coordinate.

Also, we will make contact with the approach developed in some recent papers (Cicogna and Gaeta, 1990; Gaeta, 1990, 1991; Cicogna, 1990), in which bifurcations were considered in the presence of symmetry under a general group of diffeomorphisms (i.e., not necessarily linear transformations), also called Lie-point symmetries (Olver, 1986; Bluman and Kumei, 1989; Ovsjannikov, 1982), and will relate the invariance of periodic solutions under such symmetries to topological invariance of the corresponding braids and therefore knots.

It should be remarked that in the above-mentioned approach to bifurcations of dynamical systems, a Hopf bifurcation is also seen as a modification in the topology of the trajectory. This is the simplest example of a subject which we think is worth being explored, i.e., the relations among bifurcations and modifications in the topology of solutions.

2. FORCED PERIODIC OSCILLATIONS

We want to study forced oscillations of nonlinear oscillators, a topic of independent interest and central in several branches of physics. These are modeled, in the case of finite-dimensional systems, by equations of the type

$$\ddot{u} = F(u, \dot{u}) + \varepsilon f(t) \tag{1}$$

where $f(t)$, the forcing term, satisfies

$$f(t+T) = f(t) \qquad \forall t \tag{2}$$

$$\langle f(t) \rangle \equiv \frac{1}{T} \int_0^T f(t)\, dt = 0 \tag{3}$$

and $u \in R^N$, f: $R \to R^N$.

One is interested, in particular, in the periodic solutions of (1), i.e., in solutions $u = \phi(t)$ such that for some $\tilde{T}$

$$\phi(t + \tilde{T}) = \phi(t) \quad \forall t \tag{4}$$

In this case, the period $\tilde{T}$ will be a multiple of the forcing period T:

$$\tilde{T} = NT; \qquad N = 0, 1, 2, \ldots \tag{5}$$

Periodic solutions have topological properties, and the purpose of this paper is to explore these, and show how they are related to braid invariants.

Equation (1) is nonautonomous, so that we should study it in the enlarged phase space $P = M \times T$, where $M = \{(u, \dot{u})\}$ and T is the R^1 space corresponding to the time coordinate. We can also study the equivalent first-order system

$$\dot{u} = v, \qquad \dot{v} = F(u, v) + \varepsilon f(t) \tag{6}$$

In dynamical systems theory, one also considers the associated system

$$\dot{u} = v, \qquad \dot{v} = F(u, v) + \varepsilon f(t), \qquad \dot{t} = 1$$

called the suspension of the system (6).

Now $M = \{(u, v)\}$ and it is natural to look at P as a fiber bundle with base T and fiber $M = R^{2N}$ on which a connection A has been defined, to give the covariant derivative

$$D_t = \partial_t + v\partial_u + [F(u, v) + \varepsilon f(t)]\partial_v \tag{7}$$

which is nothing else than the Lie derivative under the flow of (6). If now we put $Z = (u, v; t) \in R^{2N+1}$, equation (6) is written as

$$\dot{Z} = D_t Z \tag{8}$$

We have a natural (Poincaré) map

$$Z_{n+1} = B \cdot Z_n \tag{9}$$

corresponding to

$$Z(T) = Z(0) + \int_0^T \dot{Z}(t)\, dt \tag{10}$$

The periodic orbits correspond to points Z_0 such that $\exists N \geq 0$: $B^N Z_0 = Z_0$, N being the same as in (5).

Actually, we can consider $\tilde{P}$ instead of P,

$$\tilde{P} = M \times S^1 \equiv P/Z \tag{11}$$

and the periodic orbits are characterized by the fact that they correspond to closed curves: i.e., while a generic orbit $\omega = \{u(t), t \in R\}$ is $\omega \simeq R^1$, for a periodic orbit ω_p one has $\omega_p \simeq S^1$. So, we can see periodic orbits as being characterized by a topological property.

Now, one would also like to distinguish among periodic orbits on the basis of finer topological properties. We have seen before that the (minimal) period of an orbit can be seen as a topological property (this corresponds to the *winding number*); therefore, we are actually asking if there are topological invariants which distinguish among isoperiodic orbits. Actually, for $N = 1$ one has a (highly) nontrivial topological invariant polynomial attached to any periodic orbit, by the simple construction which we present now.

3. TOPOLOGICAL INVARIANCE OF PERIODIC OSCILLATIONS

Let us see $\tilde{P}$ as $M \times [0, 1]$ with $M_0 = M \times \{0\}$ and $M_1 = M \times \{1\}$ identified. Then an N-periodic solution $\phi(t)$ [i.e., such that $\phi(t + Nt) = \phi(t)$] will be characterized by N curves (sections of the bundle $\tilde{P}$) joining M_0 and M_1; these are distinct if NT is the minimal period of ϕ [i.e., $\phi(t + KT) \neq \phi(t)$ for $K < N$]. Moreover, these curves $\gamma_1(t), \ldots, \gamma_N(t)$ hit M_0 and M_1 in sets of equivalent points, i.e., the sets

$$\Gamma_0 = \{\gamma_1(0), \ldots, \gamma_N(0)\} \rightarrow M_0$$

$$\Gamma_1 = \{\gamma_1(1), \ldots, \gamma_N(1)\} \rightarrow M_1$$

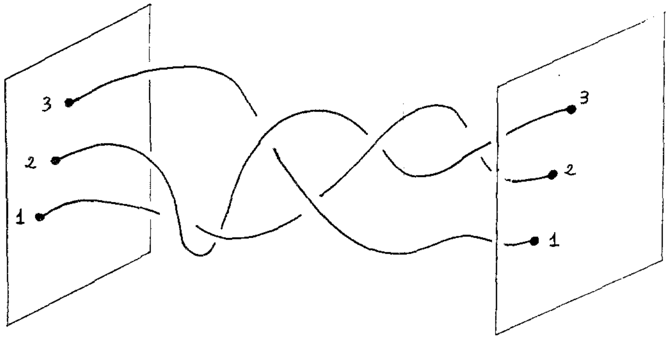

Fig. 1. An N-periodic solution can be seen as an N-braid.

are the same set in $\tilde{P}$, or under the identification of M_0 and M_1 (see Figure 1).

Such a set of curves in $M \times [0, 1]$ is known as an *N-braid*; it is transformed by the identification of M_0 and M_1 into a *knot* (see Figure 2), which is actually the periodic solution in the space $\tilde{P}$. The set of N-braids is a group, denoted B_N.

The problem of finding topological invariants of braids or knots (the two are equivalent) is an old one in topology. At present, our most powerful tool is the *Jones polynomial*, or variations of it.

A first point to make clear is that "topological invariance" is the same for periodic solutions and for braids.

Let us go back to equation (6). We can operate on P by Diff$(M \times R)$ and determine the vector field $\eta \in$ Diff$(M \times R)$ which transforms solutions

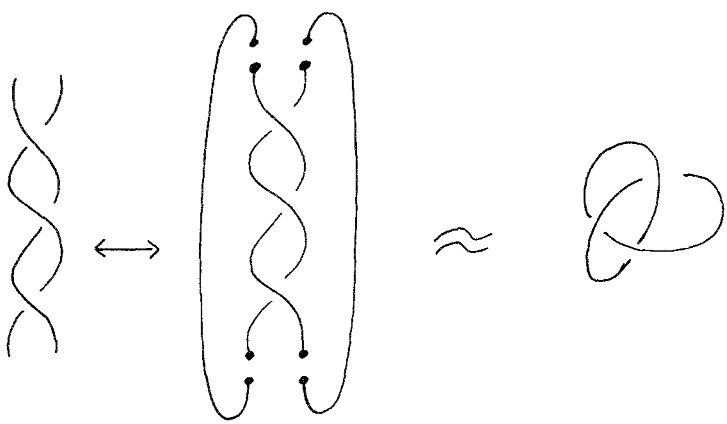

Fig. 2. Any braid can be transformed in a canonical way into a knot; this is shown here for the threefoil knot.

into solutions; these form the *symmetry algebra* $\mathcal{G}_\Delta$ of equation (6) (Olver, 1986; Bluman and Kumei, 1989; Ovsjannikov, 1982), which generates its symmetry group G_Δ. Its determination is algorithmic and is explained, e.g., in Olver (1986), Bluman and Kumei (1989), and Ovsjannikov (1982). Actually, this is too general: in fact, the problem has a natural time scale T, and we should not alter it. Another way of seeing the same point is that we can always put, once and for all, $T=1$. This means that, if

$$\eta=\phi(u, v, t)\partial_u+\psi(u, v, t)\partial_v+\tau(u, v, t)\partial_t \tag{12}$$

is a generic vector field in $\mathrm{Diff}(M\times R)$, we should ask that

$$\tau(u, v, 0)=\tau(u, v, T)=0 \tag{13}$$

or, equivalently,

$$[\eta]_{\partial\tilde{P}}=\eta_0\in\mathrm{Diff}(M) \tag{14}$$

Notice that η must be the same on M_0 and M_1, since they are identified.

There are also, in $\mathrm{Diff}(M\times R)$, transformations which are not interesting: in particular, if we just operate by a rigid (global in the gauge-theoretic language) transformation, we surely obtain something equivalent, so we finally should consider on P only the diffeomorphisms in

$$\mathrm{Diff}(M\times R)/\mathrm{Diff}(M) \tag{15}$$

and on $\tilde{P}$ those in

$$\mathrm{Diff}(M\times S^1)/\mathrm{Diff}(M) \tag{16}$$

with moreover the condition (14): i.e., we consider

$$\mathcal{G}^0_\Delta=\{\eta\in\mathrm{Diff}(M\times S^1)/\mathrm{Diff}(M):\tau(u, v, 0)=0\} \tag{17}$$

Notice that we could have taken any other t_0 as reference point for $S^1=R^1/Z$: i.e., we should ask $\tau=0$ *tout court*, so that $\eta=\phi(u, v, t)\partial_u+\psi(u, v, t)\partial_v$, with $\phi_t^2+\psi_t^2$ not identically 0. This corresponds to asking for *vertical* vector fields in gauge-theoretic langauge, i.e., for *fiber-preserving* diffeomorphisms: each fiber of the bundle is invariant under such diffeomorphisms.

This is exactly the class of admitted transformations of braids, so we have equivalent concepts of "topological invariance."

4. CONCLUSIONS AND OUTLOOK

Summarizing, our construction gives the following result:

Given equation (1), its N-periodic solutions have a natural representation in terms of N-braids; in this way a Jones polynomial can be associated to any periodic solution, and two solutions with different Jones polynomials are necessarily topologically distinct; in particular, they cannot be transformed one into the other by the action of G_Δ.

At this point, some questions arise naturally:

1. How can this result help in the actual computation of periodic solutions?
2. Given an equation (i.e., a nonlinear oscillator), is there any way to know which kind of $b \in B_n$ will be represented by its n-periodic solutions?
3. If a solution corresponding to a braid $b \in B_n$ undergoes a bifurcation (i.e., a period doubling), which kind of braids b' will correspond to the bifurcating solutions?

We hope to be able to progress toward an answer to these questions in a future publication.

NOTE ADDED IN PROOF

After the completion of this work, the author became aware that question (3) above has been considered by Crawford and Omohundro, *Physica D*, **13** (1984), 161–180. He thanks Prof. Crawford for pointing out this paper.

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